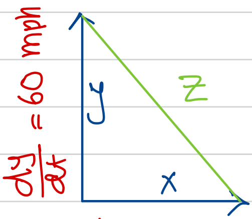
1. Two cars start moving from the same point. One travels north at 60 mph and the other travels east at 25 mph. At what rate is the distance between the cars increasing two hours later?



Want to find $\frac{dz}{dt}$.

By pythagorean theorem:

$$x^2 + y^2 = z^2 \quad (1)$$

$$\frac{dx}{dt} = 25 \text{ mph}$$

Note that Velocity $= \frac{\text{distance}}{\text{time}}$

$\Rightarrow$ to find x:

$$25 = \frac{x}{2} \Rightarrow x = 50 \quad (2)$$

to find y:

$$60 = \frac{y}{2} \Rightarrow y = 120 \quad (3)$$

(1), (2), (3) $\Rightarrow$ $z^2 = 50^2 + 120^2 = 16900$

$\Rightarrow z = 130$

Implicit differentiation on $z^2 = x^2 + y^2$:

$$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2 + y^2)$$

$$\Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

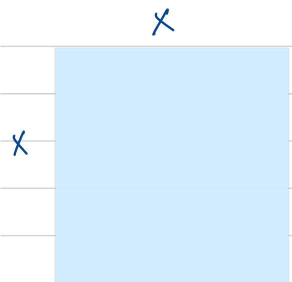
$$= \frac{1}{130}(50 \times 25 + 120 \times 60)$$

$$= \frac{1}{130}(1250 + 7200) = 65$$

$$\Rightarrow \frac{dz}{dt} = 65 \text{ mph}$$

2. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2 ?

$\frac{dx}{dt}$



Area of a square: $A(x) = x^2$ (1)

Want to find $\frac{dA}{dt}$

$A(x) = 16 = x^2 \Rightarrow x = 4$

Implicit differentiation with respect to t on (1):

$$\frac{d}{dt} A = \frac{d}{dt}(x^2) = 2x \frac{dx}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2 \times 4 \times 6 = 48 \ \text{cm}^2/\text{s}$$

3. Find the linearization of $g(x) = \dfrac{1}{\sqrt{x}}$ near $x = 4$.

Same as "linear approximation" and finding the "tangent line".

equation of a line: $y - y_0 = m(x - x_0)$

equation of a tangent line: $g(x) - g(x_0) = g'(x_0)(x - x_0)$ I

$$g(x) = \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$$

$$\Rightarrow g'(x) = -\frac{1}{2} x^{-3/2} = \frac{-1}{2(\sqrt{x})^3}$$

$$x_0 = 4 \Rightarrow \begin{cases} g(x_0) = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2} \\ g'(x_0) = \dfrac{-1}{2(\sqrt{4})^3} = \dfrac{-1}{16} \end{cases}$$

Replacing in I: $g(x) - \dfrac{1}{2} = \dfrac{-1}{16}(x - 4)$

4. Find the absolute maximum and the absolute minimum values of the indicated function on the given interval:

(a) $f(x) = \dfrac{\ln x}{x^2}$ on $\left[\dfrac{1}{2}, 4\right]$,

(b) $f(t) = \dfrac{\sqrt{t}}{1+t^2}$ on $[0, 2]$,

(c) $f(x) = xe^{x/2}$ on $[-3, 1]$,

(d) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$.

a) $f(x) = \dfrac{\ln x}{x^2}$ on $[1/2, 4]$

$$f'(x) = \frac{\frac{1}{x} \cdot x^2 - 2x\ln x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1-2\ln x}{x^3}$$

$$f'(x) = 0 \Rightarrow 1 - 2\ln x = 0 \Rightarrow \ln x = \frac{1}{2} \Rightarrow e^{\ln x} = e^{1/2}$$

$$\Rightarrow x = e^{1/2} = \sqrt{e}$$ potential critical point

$$f(\sqrt{e}) = \frac{\ln(\sqrt{e})}{(\sqrt{e})^2} = \frac{\frac{1}{2}}{e} = \frac{1}{2e} \rightarrow \text{max}$$

$$f\left(\tfrac{1}{2}\right) = \frac{\ln(1/2)}{\left(\frac{1}{2}\right)^2} = \frac{-\ln(2)}{\frac{1}{4}} = -4\ln(2) \rightarrow \text{min}$$

$$f(4) = \frac{\ln(4)}{4^2} = \frac{2\ln(2)}{16} = \frac{\ln(2)}{8}$$

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b) $f(t) = \dfrac{\sqrt{t}}{1+t^2} = \dfrac{t^{1/2}}{1+t^2}$ on $[0, 2]$

$$f'(t) = \frac{\frac{1}{2}t^{-1/2}(1+t^2) - 2t\, t^{1/2}}{(1+t^2)^2} = \frac{\frac{(1+t^2)}{2\sqrt{t}} - 2t\sqrt{t}}{(1+t^2)^2}$$

$$= \frac{1+t^2 - 4t^2}{2\sqrt{t}(1+t^2)^2} = \frac{1-3t^2}{2\sqrt{t}\,(1+t^2)^2}$$

$$f'(t) = 0 \Rightarrow 1 - 3t^2 = 0 \Rightarrow t^2 = \frac{1}{3} \Rightarrow \begin{cases} t = \frac{1}{\sqrt{3}} \\ \cancel{t = -\frac{1}{\sqrt{3}}} \end{cases}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{\sqrt{\frac{1}{\sqrt{3}}}}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{3^{-1/4}}{1+\frac{1}{3}} = \frac{3^{-\frac{1}{4}}}{\frac{4}{3}} = \frac{3^{\frac{3}{4}}}{4}$$ max

$f(0) = 0$ min

$$f(2) = \frac{\sqrt{2}}{1+4} = \frac{\sqrt{2}}{5}$$

c) $f(x) = x e^{x/2}$ on $[-3, 1]$

$$f'(x) = e^{x/2} + \frac{1}{2} x e^{x/2}$$

$$f'(x) = 0 \Rightarrow e^{x/2}\left(1 + \frac{1}{2}x\right) = 0 \Rightarrow x = -2$$

$f(-2) = -2e^{\frac{-2}{2}} = \frac{-2}{e}$ min

$f(-3) = -3e^{-3/2} = \frac{-3}{(\sqrt{e})^3}$

$f(1) = \sqrt{e}$ max

d) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$

$f'(x) = 12x^3 - 12x^2 - 24x$

$$f'(x) = 0 \Rightarrow 12x(x^2 - x - 2) = 0 \Rightarrow 12x(x-2)(x+1) = 0 \Rightarrow \begin{cases} x = 0 \\ x = 2 \\ x = -1 \end{cases}$$

$f(-1) = -4$

$f(-2) = 33$ max

$f(0) = 1$

$f(2) = -31$ min

$f(3) = 28$

5. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

(a) $f(x) = 2x^2 - 3x + 1$, $[0, 2]$

(b) $f(x) = \frac{1}{x}$, $[1, 4]$

Need to know: "MVT": $f(x)$ continuous & diff on $[a,b]$ $\Rightarrow$ there is a $c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$

a) $f(x) = 2x^2 - 3x + 1$ on $[0,2]$

$f'(x) = 4x - 3$

$f'(c) = 4c - 3 = \frac{f(2) - f(0)}{2} = \frac{3-1}{2} = 1$

$\Rightarrow 4c - 3 = 1 \Rightarrow 4c = 4 \Rightarrow c = 1 \in [0,2]$ ✓

Make sure c is in the given interval.

b) $f(x) = \frac{1}{x}$ on $[1,4]$

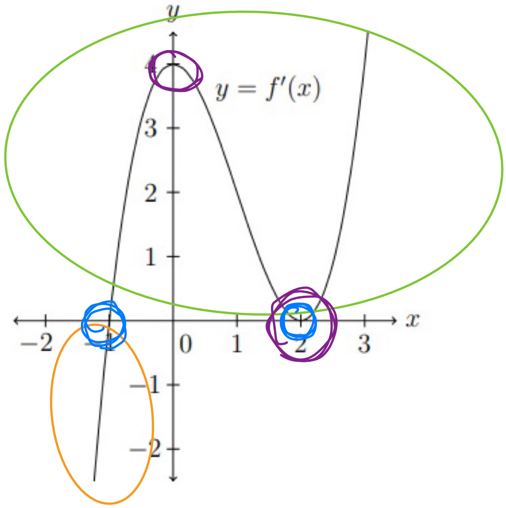
$f'(x) = \frac{-1}{x^2}$

$f'(c) = \frac{-1}{c^2} = \frac{f(4) - f(1)}{4-1} = \frac{\frac{1}{4} - 1}{3} = \frac{\frac{-3}{4}}{3} = \frac{-1}{4}$

$\Rightarrow \frac{-1}{c^2} = \frac{-1}{4} \Rightarrow c^2 = 4 \Rightarrow \begin{cases} c = 2 \in [1,4] ✓ \\ \cancel{c = -2} \end{cases}$

not on $[1,4]$

6. Below is a graph of $y = f'(x)$. Determine the intervals where $f(x)$ is increasing and decreasing on $[-1, 3]$, the x-values where $f(x)$ has local maxima and minima, and the x-values where $f(x)$ has inflection points.



- $f'(x) < 0$ on $(-2, -1)$ $\Rightarrow$ $f(x)$ decreasing on $(-2, -1)$
- $f'(x) > 0$ on $(-1, 3)$ $\Rightarrow$ $f(x)$ increasing on $(-1, 3)$
- $f'(x) = 0$ at $x = -1$ and $x = 2$:

  $x = -1$ is a local min (f decreasing to -1, then increasing)

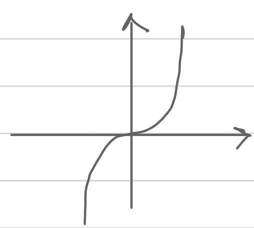
  ① $x = 2$ is not a min or max (f increasing all the time)

  * To better understand ①, take a look at $f(x) = x^3$

  $f'(x) = 3x^2$

  $f'(x) = 0$ at $x = 0$

  but increasing everywhere



- $f'(x)$ has local max at $x = 0$ & local min at $x = 2$

$\Rightarrow f''(x) = 0$ at $x = 0$ & $x = 2$

$f'(x)$ increasing on $(-2, 0) \cup (2, 3) \Rightarrow f''(x) > 0$

$f'(x)$ decreasing on $(0, 2) \Rightarrow f''(x) < 0$

x	-2 to 0	0 to 2	2 to 3
$f'(x)$	$\nearrow$	$\searrow$	$\nearrow$
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cup$

inflection point at $x = 0$

inflection point at $x = 2$

7. Find the intervals on which the indicated function is increasing or decreasing, find the local maximum and minimum of the function, and find the intervals of concavity and inflection points.

   (a) $f(x) = x^3 - 3x^2 - 9x + 4$
   (b) $f(x) = 2x^3 - 9x^2 + 12x - 3$
   (c) $f(x) = x^4 - 2x^2 + 3$
   (d) $f(x) = e^{2x} + e^{-x}$

Need to know:

$$\begin{cases} f'(x) > 0 \Rightarrow f \text{ increasing} \\ f'(x) < 0 \Rightarrow f \text{ decreasing} \\ f'(x) = 0 \Rightarrow \text{potential critical point} \end{cases}$$

$$\begin{cases} f''(x) > 0 \Rightarrow \text{concave up} \\ f''(x) < 0 \Rightarrow \text{concave down} \\ f''(x) = 0 \Rightarrow \text{potential inflection point} \end{cases}$$

a) $f(x) = x^3 - 3x^2 - 9x + 4$

$f'(x) = 3x^2 - 6x - 9$

$f'(x) = 0 \Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = 3$ or $x = -1$

$f''(x) = 6x - 6$

$f''(x) = 0 \Rightarrow 6x - 6 = 0$

$\Rightarrow x = 1$

x	$-\infty$		-1		1		3		$+\infty$
$f'(x)$		$+$		$-$				$+$	
$f(x)$		$\nearrow$	max at $x=-1$	$\searrow$			min at $x=3$	$\nearrow$	
$f''(x)$		$-$				$+$			
$f(x)$		$\cap$				$\cup$			

inflection point at $x=1$

b) $f(x) = 2x^3 - 9x^2 + 12x - 3$

$f'(x) = 6x^2 - 18x + 12$

$f'(x) = 0 \Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x-2)(x-1) = 0$

$\Rightarrow x = 2$ or $x = 1$

$f''(x) = 12x - 18$

$f''(x) = 0 \Rightarrow 12x - 18 = 0$

$\Rightarrow x = \frac{3}{2}$

x	$-\infty$		1		$3/2$		2		$+\infty$
$f'(x)$		$+$			$-$			$+$	
$f(x)$		$\nearrow$	max at $x=1$	$\searrow$			min at $x=2$	$\nearrow$	
$f''(x)$			$-$				$+$		
$f(x)$			$\cap$				$\cup$		

inflection point at $x = 1.5$

c) $f(x) = x^4 - 2x^2 + 3$

$f'(x) = 4x^3 - 4x$

$f'(x) = 0 \Rightarrow 4x(x^2 - 1) = 0$

$\Rightarrow x = 0,\ x = 1,\ x = -1$

$f''(x) = 12x^2 - 4$

$f''(x) = 0 \Rightarrow 12x^2 - 4 = 0$

$\Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

x	$-\infty$		-1		$-\frac{1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$		1		$+\infty$
$f'(x)$		$-$			$+$				$-$			$+$	
$f(x)$		$\searrow$	min		$\nearrow$		max		$\searrow$		min	$\nearrow$	
$f''(x)$			$+$				$-$				$+$		
$f(x)$			$\cup$				$\cap$				$\cup$		

inflection point at $x = -\frac{1}{\sqrt{3}}$

inflection point at $x = \frac{1}{\sqrt{3}}$

d) $f(x) = e^{2x} + e^{-x}$

$f'(x) = 2e^{2x} - e^{-x}$

$f'(x) = 0 \Rightarrow e^{-x}(2e^{3x} - 1) = 0$

$\Rightarrow 2e^{3x} - 1 = 0$

$\Rightarrow e^{3x} = \frac{1}{2}$

$\Rightarrow 3x = -\ln(2)$

$\Rightarrow x = \frac{-1}{3}\ln(2)$

$f''(x) = 4e^{2x} + e^{-x}$

$f''(x) = 0 \Rightarrow e^{-x}(4e^{3x} + 1) = 0$

$\Rightarrow 4e^{3x} + 1 = 0$

$\Rightarrow e^{3x} = \frac{-1}{4}$

$\Rightarrow 3x = \ln(-1/4)$

ln doesn't take negative values

$\Rightarrow$ No inflection points

x	$-\infty$	-1	$-1/3 \ln(2)$	0	$+\infty$
$f'(x)$		$-$		$+$	
$f(x)$		$\searrow$	min	$\nearrow$	
$f''(x)$			$+$		
$f(x)$			$\cup$		

If you're asked to write everything in terms of intervals, using the table:

- $f(x)$ is decreasing on $(-\infty, \frac{-1}{3}\ln(2))$
- $f(x)$ is increasing on $(\frac{-1}{3}\ln(2), +\infty)$
- $f(x)$ has a local minimum at $x = \frac{-1}{3}\ln(2)$

- $f(x)$ is concaving up on $(-\infty, +\infty)$
- $f(x)$ has no inflection points.

8. Find the limits using l'Hospital's Rule. If l'Hospital's Rule doesn't apply, explain why.

(a) $\displaystyle\lim_{x\to 0} \frac{\tan(3x)}{\sin(2x)}$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

(c) $\displaystyle\lim_{x\to\infty} x^3 e^{-x^2}$

(d) $\displaystyle\lim_{x\to 1} \frac{x^8 - 1}{x^5 - 1}$

a) $\displaystyle \overset{0/0}{\lim_{x\to 0}} \frac{\tan(3x)}{\sin(2x)} \overset{L'H}{=} \lim_{x\to 0} \frac{3\sec^2(3x)}{2\cos(2x)} = \frac{3}{2}$

b) $\displaystyle \overset{0/0}{\lim_{x\to 0}} \frac{e^x - 1 - x}{x^2} \overset{L'H}{=} \overset{0/0}{\lim_{x\to 0}} \frac{e^x - 1}{2x} \overset{L'H}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$

c) $\displaystyle \lim_{x\to\infty} x^3 e^{-x^2} = \overset{\infty/\infty}{\lim_{x\to\infty}} \frac{x^3}{e^{x^2}} \overset{L'H}{=} \lim_{x\to\infty} \frac{3x^2}{2x e^{x^2}}$

$\displaystyle = \overset{\infty/\infty}{\lim_{x\to\infty}} \frac{3x}{2e^{x^2}}$

$\displaystyle \overset{L'H}{=} \lim_{x\to\infty} \frac{3}{4x e^{x^2}} = 0$

d) $\displaystyle \overset{0/0}{\lim_{x\to 1}} \frac{x^8 - 1}{x^5 - 1} \overset{L'H}{=} \lim_{x\to 1} \frac{8x^7}{5x^4} = \frac{8}{5}$

9. Find two numbers whose difference is 100 and whose product is minimum.

$$a - b = 100 \Rightarrow a = 100 + b$$

$$P = ab = (100+b)b = 100b + b^2$$

$$P' = 100 + 2b = 0 \Rightarrow b = -50$$

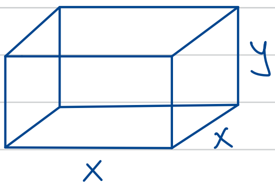
$$\Rightarrow a = 100 + b = 100 - 50 = 50$$

$$\Rightarrow P = 50 \times (-50) = -2500$$

Justifying that $b=-50$ gives the min:

	$-\infty$		-50		∞
P'		$-$		$+$	
P		$\searrow$	min	$\nearrow$	

10. If 12 m^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.



$$\text{Area} = x^2 + 4xy = 12$$

$$\Rightarrow 4xy = 12 - x^2$$

$$\Rightarrow y = \frac{12 - x^2}{4x}$$

$$V = x^2 y = x^2\left(\frac{12 - x^2}{4x}\right) = \frac{1}{4}(12x - x^3)$$

$$V' = \frac{1}{4}(12 - 3x^2) = 0$$

$$\Rightarrow 12 - 3x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

$$x = 2 \Rightarrow y = \frac{12-4}{8} = 1$$

$$\Rightarrow V = 2^2 \times 1 = 4$$

Justifying that $x = 2$ gives the max:

	0	$x=2$	$+\infty$
V'	$+$		$-$
V	$\nearrow$	max	$\searrow$

11. Use Newton's method to approximate $\sqrt[4]{74}$ using $x_1 = 3$. Note that in Newton'w method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Let $\sqrt[4]{74} = x \implies (\sqrt[4]{74})^4 = (x)^4$

$\implies 74 = x^4$

$\implies x^4 - 74 = 0$

$\implies f(x) = x^4 - 74 \quad \rightarrow f(3) = 81 - 74 = 7$

$f'(x) = 4x^3 \quad \rightarrow f'(3) = 4 \times 27 = 108$

$$\implies \sqrt{74} \simeq 3 - \frac{7}{108} = \frac{317}{108}$$

12. Use Newton's method with $x_1 = -1$ to find x_2, the second approximation to the root of $2x^3 - 3x^2 + 2 = 0$.

$f(x) = 2x^3 - 3x^2 + 2 \rightarrow f(-1) = -3$

$f'(x) = 6x^2 - 6x \rightarrow f'(-1) = 12$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_2 = -1 - \frac{-3}{12} = -1 + \frac{1}{4} = \frac{-3}{4}$$