1. (**Integration by Parts**) Evaluate the integral.

   (a) $\int x \cos 5x \mathrm{d}x$

   (b) $\int \arctan x \mathrm{d}x$

   (c) $\int (\ln x)^2 \mathrm{d}x$

   (d) $\int_0^1 (x^2+1)e^{-x} \mathrm{d}x$

   (e) $\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) \mathrm{d}\theta$

2. (**Trigonometric Integrals**) Evaluate the integral.

   (a) $\int \sin^5(2t)\cos^2(2t)\mathrm{d}t$

   (b) $\int \frac{1-\tan^2(x)}{\sec^2(x)}\mathrm{d}x = \int \frac{1-\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}\, dx = \int \frac{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}{1/\cos^2 x}\, dx$

   (c) $\int \underbrace{\tan(x)\sec^3(x)}_{\tan(x)\cdot\sec(x)\ \cdot\ \sec^2(x)}\mathrm{d}x$

   $= \int (\cos^2 x - \sin^2 x)\, dx$

   (d) $\int \sin(x)\underbrace{\sec^5(x)}_{\sec\ \cdot\ \sec^4}\mathrm{d}x$

$= \int \cos(2x)\, dx$

tan

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$= \frac{1}{2}\sin(2x) + C$

c) $\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^3 x}\, dx = \int \frac{\sin x}{\cos^4 x}\, dx = \frac{1}{3\cos^3 x} + C$

$u = \sec(x)$

$du = \sec(x)\tan(x)$

$\int \underbrace{\sec^2(x)}_{u^2} \cdot \underbrace{\tan(x)\sec(x)\, dx}_{du}$

$= \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C$

$\int \frac{\sin x}{\cos^5 x}\, dx = \frac{1}{4\cos^4 x} + C = \frac{1}{4}\sec^4 x + C$

$$\int \sin x \sec^5 x \, dx = \int \underbrace{\sin x \; \overset{\frac{1}{\cos x}}{\sec x}} \sec^4 x \, dx$$

$$= \int \frac{\sin x}{\cos x} \sec^4 x \, dx$$

$$= \int \tan x \sec x \cdot \sec^3 x \, dx$$

LIPET → trig

log ↙ ↓ inverse trig ↘ Polynomials → exp

1(b) $\int \overset{1}{\arctan(x)}\,dx$

$= \arctan x \cdot x - \int \underbrace{\frac{x}{1+x^2}\,dx}_{u}$ $\frac{1}{2}du$

$= \arctan(x) \cdot x - \int \frac{1}{2}\frac{1}{u}\,du$

$= \arctan(x)\cdot x - \frac{1}{2}\ln|u| + C$

$= \arctan(x)\cdot x - \frac{1}{2}\ln|1+x^2| + C$

$u = \underline{\arctan x} \rightarrow du = \frac{1}{1+x^2}dx$

$dx = dv \rightarrow x = v$

$\int u\,dv = uv - \int v\,du$

$1+x^2 = u \rightarrow 2x\,dx = du$

1(c) $\int (\ln x)^2\,dx =$

$(\ln(x))^2 \cdot x - \int \cancel{x} \cdot 2\ln(x) \cdot \frac{1}{\cancel{x}}\,dx =$

$(\ln x)^2 \cdot x - \int 2\,\underbrace{\ln(x)}_{u}\,\underbrace{dx}_{dv} =$

$(\ln x)^2 \cdot x - 2\left(\ln(x)\cdot x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx\right) =$

$(\ln x)^2 \cdot x - 2(\ln(x)\cdot x - x) + C$

$\int 2\ln x\,dx = 2\int \ln x\,dx$

LIPET $\int u\,dv = uv - \int v\,du$

$u = (\ln x)^2 \rightarrow du = \underbrace{2\ln(x)\cdot\frac{1}{x}}$

$dv = dx \rightarrow v = x$

$(f(g(x)))' = f'(g(x))\cdot g'(x)$

$f(x) = x^2$

$g(x) = \ln(x)$

$f(g(x))$

$u = \ln(x) \rightarrow du = \underline{\frac{1}{x}dx}$

$dv = dx \rightarrow \underline{v = x}$

$\int u\,dv = uv - \int v\,du$

LIPET

$$\int \underbrace{e^x}_{\text{exp}} \underbrace{\cos x}_{\text{trig}}\, dx =$$

$u = e^x \to du = e^x dx$

$dv = \cos x\, dx \to v = \sin x$

$$e^x \cdot \sin x - \int \underbrace{\sin x}_{v}\, \underbrace{e^x}_{u}\, dx =$$

$u = e^x \to du = e^x dx$

$dv = \sin x\, dx \to v = -\cos x$

$$e^x \cdot \sin x - \left(-\cos x\, e^x - \int -\cos x\, e^x dx\right) =$$

$$\underline{e^x \cdot \sin x + \cos x\, e^x} - \underline{\int \cos x\, e^x dx}$$

$$\Rightarrow 2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x$$

$$\Rightarrow \int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2}$$

$$2(a) = \int \sin^5(2t)\cos^2(2t)\,dt =$$

$u = \cos(2t)$

$du = -2\sin(2t)\,dt$

$$\int \underbrace{\sin^4(2t)}_{(\sin^2(2t))^2}\cos^2(2t)\sin(2t)\,dt =$$

$$\int (1-\cos^2(2t))^2\cos^2(2t)\overbrace{\sin(2t)\,dt}^{-\frac{1}{2}du} =$$

$$\int (1-u^2)^2 u^2 \left(-\frac{1}{2}du\right) =$$

$$-\frac{1}{2}\int (1+u^4-2u^2)\,u^2\,du =$$

$$-\frac{1}{2}\int (u^2+u^6-2u^4)\,du = -\frac{1}{2}\left(\frac{u^3}{3}+\frac{u^7}{7}-2\frac{u^5}{5}\right)+C$$

$$= -\frac{1}{2}\left(\frac{\cos^3(2t)}{3}+\frac{\cos^7(2t)}{7}-\frac{2}{5}\cos^5(2t)\right)+C$$

$$2(c)\ \int \tan(x)\sec^3(x)\,dx =$$

$(\sec x)' = \sec x \cdot \tan x$

$$\int \overbrace{(\tan(x)\cdot\sec(x))\,dx}^{du}\ \underbrace{\sec^2(x)}_{u^2}$$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$$= \int u^2\,du = \frac{u^3}{3}+C = \frac{\sec(x)^3}{3}+C$$

3. (**Trigonometric Substitution**) Evaluate the integral.

   (a) $\displaystyle\int_2^3 \frac{\mathrm{d}x}{(x^2-1)^{3/2}}$

   (b) $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\mathrm{d}x$

   (c) $\displaystyle\int x\sqrt{1-x^4}\mathrm{d}x$

   (d) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\mathrm{d}x$

4. (**Integration of Rational Functions by Partial Fractions**) Evaluate the integral.

   (a) $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\mathrm{d}x$

   (b) $\displaystyle\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\mathrm{d}x$

   (c) $\displaystyle\int \frac{\mathrm{d}x}{x^2+x\sqrt{x}}$

   (d) $\displaystyle\int \frac{\mathrm{d}x}{1+e^x}$

3(b) $\int \frac{\sqrt{1+x^2}}{x}\,dx =$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}\sec^2\theta\, d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\, d\theta$$

$$= \int \left(\frac{\sec\theta}{\tan\theta} + \sec\theta\tan\theta\right) d\theta$$

$\csc\theta$

$$= \int \left(\frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} + \sec\theta\tan\theta\right) d\theta = \int\left(\frac{1}{\sin\theta} + \sec\theta\tan\theta\right) d\theta$$

$\ln|\csc\theta - \cot\theta| + \sec\theta + C$

$$\int \csc\theta\, d\theta = \int \frac{\csc\theta(\csc\theta - \cot\theta)}{\csc\theta - \cot\theta}\, d\theta$$

$$= \int \frac{\csc^2\theta - \csc\theta\cot\theta}{\underbrace{\csc\theta - \cot\theta}_{u}}\, d\theta$$

$du = (-\csc\theta\cot\theta + \csc^2\theta)\, d\theta$

d) $\int \frac{x}{\sqrt{1+x^2}}\, dx$ $\qquad u = 1+x^2 \longrightarrow du = 2x\,dx$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\, du$$

$$= \frac{1}{2}\left(2u^{1/2}\right)$$

$$= \sqrt{1+x^2}$$

3(a) $\displaystyle\int_2^3 \frac{dx}{(x^2-1)^{3/2}} = \int_{2}^{3} \frac{dx}{\left(\sqrt{x^2-1}\,\right)^3}$

$x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta\, d\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$x = 2 \longrightarrow \theta = \sec^{-1}2$

$x = 3 \longrightarrow \theta = \sec^{-1}3$

$$\int_{\sec^{-1}2}^{\sec^{-1}3} \frac{\sec\theta\tan\theta\, d\theta}{\left(\sqrt{\tan^2\theta}\,\right)^3} = \int_{\sec^{-1}2}^{\sec^{-1}3} \frac{\sec\theta\tan\theta\, d\theta}{(\tan\theta)^3}$$

$$= \int_{\sec^{-1}2}^{\sec^{-1}3} \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$\sec\theta = \frac{1}{\cos\theta}$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$

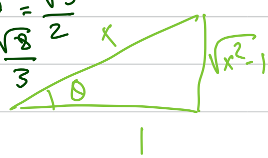
$$= \int_{\sec^{-1}2}^{\sec^{-1}3} \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\, d\theta = \int_{\sec^{-1}2}^{\sec^{-1}3} \frac{\overbrace{\cos\theta\, d\theta}^{du}}{\sin^2\theta}$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$x = 2 \longrightarrow \sin\theta = \frac{\sqrt{3}}{2}$

$x = 3 \longrightarrow \sin\theta = \frac{\sqrt{8}}{3}$



$\sin\theta = \frac{\sqrt{x^2-1}}{x}$

$$= \int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{8}}{3}} \frac{1}{u^2}\, du$$

$$= \frac{-1}{u} \Bigg|_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{8}}{3}}$$

$$= -\frac{1}{\frac{\sqrt{8}}{3}} - \left(\frac{-1}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{-3}{\sqrt{8}} + \frac{2}{\sqrt{3}}$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

$\frac{1}{u^2} = u^{-2}$

$$\int u^{-2}\, du = \frac{u^{-2+1}}{-2+1} = \frac{u^{-1}}{-1} = \frac{-1}{u}$$

4(b) $\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}$

$\rightarrow \frac{x(3-5x)}{(3x-1)(x-1)^2} = \frac{A}{3x-1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ $\qquad x=1$, $x=\frac{1}{3}$

$\rightarrow A(x-1)^2 + B(3x-1)(x-1) + C(3x-1) = 3x-5x^2$

$x=1 \Rightarrow 2C = 3-5 = -2 \Rightarrow \boxed{C=-1}$

$x=\frac{1}{3} \Rightarrow A(\cancel{\tfrac{1}{3}-1})^2 = 3(\tfrac{1}{3}) - 5(\tfrac{1}{3})^2$ $\quad \frac{2}{3}$

$\frac{4}{9}A = 1 - \frac{5}{9} = \frac{4}{9} \Rightarrow \boxed{A=1}$

$A(x^2-2x+1) + B(3x^2-3x-x+1) + \cancel{C(3x-1)} = 3x-5x^2$

$Ax^2 + 3Bx^2 = -5x^2$

$x^2 + 3Bx^2 = -5x^2 \Rightarrow 3Bx^2 = -6x^2$

$\Rightarrow 3B = -6 \Rightarrow \boxed{B=-2}$

$\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,dx = \int_2^3 \left(\frac{1}{3x-1} + \frac{-2}{x-1} + \frac{-1}{(x-1)^2}\right) dx$

$= \frac{1}{3}\ln|3x-1| - 2\ln|x-1| + \frac{1}{x-1} \Big|_2^3$

$= -\ln 2 - \frac{1}{3}\ln 5 - \frac{1}{2}$

5. Use Trapezoidal rule, the Midpoint rule, and Simpson's rule to approximate the given integral with the specified value of n.

   (a) $\int_1^2 \sqrt{x^3 - 1}\mathrm{d}x,\ n = 10$

   (b) $\int_0^4 \ln(1 + e^x)\mathrm{d}x,\ n = 8$

6. Find the approximations $\mathrm{L_n}, \mathrm{R_n}, \mathrm{T_n}$, and $\mathrm{M_n}$ for $\mathrm{n} = 5, 10$, and 20 . Then compute the corresponding errors $\mathrm{E_L}, \mathrm{E_R}, \mathrm{E_T}$, and $\mathrm{E_M}$. (Round your answers to six decimal places. You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when n is doubled?

   (a) $\int_0^1 xe^x\mathrm{d}x$

   (b) $\int_1^2 \frac{1}{x^2}\mathrm{d}x$

5(a) $\int_1^2 \sqrt{x^3-1}\, dx$

$f(x) = \sqrt{x^3-1}$ $\quad \Delta x = \frac{2-1}{10} = 0.1$

$n = 10$ $\quad \Delta x = \frac{b-a}{n}$

$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$

$= \frac{0.1}{2}\left[f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.9) + f(2)\right]$

$M_n = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right]$; $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

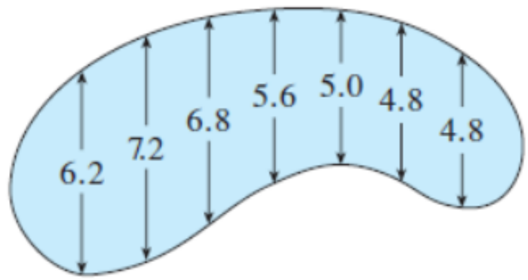
$= 0.1\left[f(1.05) + f(1.15) + f(1.25) + \cdots + f(1.95)\right]$

$\bar{x}_1 = \frac{1}{2}(x_0 + x_1) = \frac{1}{2}(1 + 1.1) = 1.05$

$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

$= \frac{0.1}{3}\left[f(1) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2)\right]$

7. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use Simpson's Rule to estimate the area of the pool.



8. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\mathrm{d}x$ $= \displaystyle\lim_{t \to \infty} \int_0^t \frac{1}{\sqrt[4]{1+x}}\,dx$

(b) $\displaystyle\int_{-\infty}^0 2^r \mathrm{d}r$

(c) $\displaystyle\int_2^\infty \frac{\mathrm{d}v}{v^2+2v-3}$

(d) $\displaystyle\int_{-1}^2 \frac{x}{(x+1)^2}\mathrm{d}x$

8(a) $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{\sqrt[4]{1+x}}\,dx$

$$= \lim_{t\to\infty}\int_0^t (1+x)^{-\frac{1}{4}}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{(1+x)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1}\right]_0^t$$

$$= \lim_{t\to\infty}\left[\frac{4}{3}(1+x)^{\frac{3}{4}}\right]_0^t = \lim_{t\to 0}\left[\frac{4}{3}(1+t)^{\frac{3}{4}} - \frac{4}{3}\right] = \infty$$

Divergent

8(b) $\int_{-\infty}^{0} 2^r \, dr = \lim_{t \to -\infty} \int_{t}^{0} 2^r \, dr$

$$= \lim_{t \to -\infty} \left[\frac{2^r}{\ln(2)} \right]_t^0$$

$$= \lim_{t \to -\infty} \left[\frac{2^0}{\ln(2)} - \frac{2^t}{\ln(2)} \right]$$

$$= \lim_{t \to -\infty} \left[\frac{1}{\ln(2)} - \cancel{\frac{2^t}{\ln(2)}} \right] = \frac{1}{\ln(2)}$$

0

Convergent

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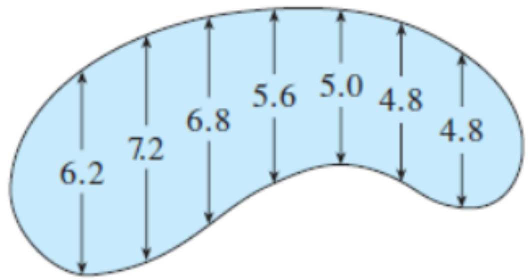

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(b) $\displaystyle\int_{-\infty}^{0} 2^r \mathrm{d}r$

(c) $\displaystyle\int_2^{\infty} \frac{\mathrm{d}v}{v^2+2v-3}$

(d) $\displaystyle\int_{-1}^{2} \frac{x}{(x+1)^2} \mathrm{d}x$

The End.