1. Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \dfrac{3+5n^2}{n+n^2}$

(b) $a_n = \ln(n+1) - \ln(n)$

(c) $a_n = \dfrac{n^2}{\sqrt{n^3+4n}}$

(d) $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \cdots\}$

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{3+5n^2}{n+n^2} = 5 \Rightarrow$ converges

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\ln(n+1) - \ln(n)\right) = \lim_{n\to\infty} \ln\left(\dfrac{n+1}{n}\right)$

$= \lim_{n\to\infty} \ln\left(1+\dfrac{1}{n}\right) = \ln(1) = 0 \Rightarrow$ converges

(c) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2}{\sqrt{n^3+4n}} = \infty \Rightarrow$ diverges

(d) Never stays arbitrarily close to any of 1 or 0
$\Rightarrow$ diverges.

2. Let $a_n = \dfrac{2n}{3n+1}$.

(a) Determine whether $\{a_n\}$ is convergent.

(b) Determine whether $\sum_{n+1}^{\infty} a_n$ is convergent.

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3} \Rightarrow$ Converges

(b) Since $\lim_{n\to\infty} a_n \neq 0$, series diverges by divergent test.

3. Determine whether the geometric series is convergent or divergent. If it is convergent, find the sum.

(a) $2 + 0.5 + 0.125 + 0.03125 + \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

(a) $r = \frac{0.5}{2} = \frac{1}{4} \Rightarrow |r| < 1$

$\Rightarrow$ converges to $\frac{a}{1-r} = \frac{2}{1-1/4} = \frac{8}{3}$

(b) $\sum \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} \underbrace{\sum \left(\frac{-3}{4}\right)^{n-1}}_{bn}$

bn converges to $\frac{a}{1-r}$ where $a=1$, $r = \frac{-3}{4}$

$\frac{a}{1-r} = \frac{1}{1-\left(\frac{-3}{4}\right)} = \frac{4}{7}$

$\Rightarrow a_n = \frac{1}{4} b_n$ converges to $\frac{1}{4} \times \frac{4}{7} = \frac{1}{7}$

4. Determine whether $s_n = \displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$ is convergent or divergent by expressing it as a telescoping sum. Find the sum if it's convergent.

$$S_n = \sum_{n=2}^{\infty} \frac{1}{n(n-1)(n+1)} = \sum_{n=2}^{\infty} \left(\frac{-1}{n} + \frac{1/2}{n-1} + \frac{1/2}{n+1} \right) = \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1} \right)$$

$$= \frac{1}{2} \Big[\left(\frac{1}{1} - \frac{2}{2} + \cancel{\frac{1}{3}} \right) + \left(\frac{1}{2} - \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{2}{4}} + \frac{1}{5} \right) + \cancel{\cdots} + \left(\frac{1}{n-3} - \frac{2}{n-2} + \cancel{\frac{1}{n-1}} \right) + \left(\frac{1}{n-2} - \cancel{\frac{2}{n-1}} + \frac{1}{n} \right) + \left(\cancel{\frac{1}{n-1}} - \frac{2}{n} + \frac{1}{n+1} \right) \Big]$$

$$= \frac{1}{2} \left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1} \right) = \frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2}$$

$$\Rightarrow \sum s_n = \lim_{n \to \infty} s_n = \frac{1}{4}$$

5. Express $0.\overline{46} = 0.46464646\cdots$ as a ratio of integers.

$$0.464646\ldots = 0.46 + 0.0046 + 0.000046 + \cdots$$
$$= \frac{46}{100} + \frac{46}{100^2} + \frac{46}{100^3} + \cdots$$
$$\Rightarrow a = \frac{46}{100}, \quad r = \frac{1}{100}$$
$$|r| < 1 \Rightarrow 0.\overline{46} \text{ converges to } \frac{a}{1-r} = \frac{46}{99}$$

6. Find the values of x for which the series $\sum_{n=1}^{\infty}(-5)^n x^n$ converges. Find the sum of the series for those values of x.

$$\sum_{n=1}^{\infty}(-5)^n x^n = \sum_{n=1}^{\infty}(-5x)^n$$

Geometric series with $r = -5x$

$$\Rightarrow a_n \text{ converges} \iff |r| < 1$$
$$\iff |-5x| < 1$$
$$\iff |x| < \frac{1}{5}$$
$$\iff -\frac{1}{5} < x < \frac{1}{5}$$
$$\Rightarrow \sum a_n = \frac{a}{1-r} = \frac{-5x}{1-(-5x)} = \frac{-5x}{1+5x}$$

7. Use the integral test to determine whether the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ is convergent or divergent.

$f(x) = x^2 e^{-x^3}$ positive, continuous, decreasing on $[1,\infty)$

$$\int_1^\infty x^2 e^{-x^3}\,dx = \lim_{t\to\infty}\int_1^t x^2 e^{-x^3}\,dx = \lim_{t\to\infty}\left[\frac{-1}{3}e^{-x^3}\right]_1^t$$

$$= \frac{-1}{3}\lim\left(e^{-t^3} - e^{-1}\right) = \frac{1}{3e}.$$

* Use u-substitution from calc I to evaluate the integral.

8. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is convergent or divergent.

$f(x) = \frac{1}{x\ln x}$ continuous, positive on $[2,\infty)$, decreasing since $f'(x) < 0$ for $x > 2$

$$\Rightarrow \int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t$$

$$= \lim_{t\to\infty}\left[\ln(\ln(t)) - \ln(\ln(2))\right]$$

$= \infty$ $\Rightarrow$ diverges.

9. Determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$

(a) $S_n = \dfrac{9^n}{3+10^n} < \dfrac{9^n}{10^n} = \left(\dfrac{9}{10}\right)^n$ & $\sum\left(\dfrac{9}{10}\right)^n$ convergent

$\Rightarrow S_n$ Convergent by comparison test.

(b) $\dfrac{\ln K}{K} < \dfrac{1}{K} \quad \forall K \geq 3$

by p-series test $\dfrac{1}{K}$ diverges $\Rightarrow \dfrac{\ln K}{K}$ diverges.

(c) $a_n = \dfrac{\sqrt[3]{K}}{\sqrt{K^3+4K+3}} < \dfrac{\sqrt[3]{K}}{\sqrt{K^3}} = \dfrac{K^{1/3}}{K^{3/2}} = \dfrac{1}{K^{7/6}}$

$\dfrac{1}{K^{7/6}}$ converges by p-test $\Rightarrow a_n$ Converges

(d) $a_n = \dfrac{4^{n+1}}{3^n-2} > \dfrac{4 \cdot 4^n}{3^n} = 4\left(\dfrac{4}{3}\right)^n$

$\left(\dfrac{4}{3}\right)^n$ is a geometric series with $|r| = \dfrac{4}{3} > 1 \Rightarrow$ diverges

$\Rightarrow a_n$ diverges.

(e) $a_n = \dfrac{n^2+n+1}{n^4+n^2}$, $b_n = \dfrac{1}{n^2} \rightarrow$ Converges

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+n+1}{n^2} = 1 > 0 \Rightarrow a_n$ absolute conv.

10. Test the series for convergence or divergence.

(a) $-\frac{2}{5}+\frac{4}{6}-\frac{6}{7}+\frac{8}{8}-\frac{10}{9}+\cdots$

(b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$

(a) $a_n = \sum (-1) \frac{2n}{n+4}$

$b_n = \frac{2n}{n+4}$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2n}{n+4} = 2 \neq 0$

$\Rightarrow$ diverges by Divergence test.

(b) $b_n = \frac{n^2}{n^3+4} > 0$ for $n \geq 1$, decreasing for $n \geq 2$ since $b_n' < 0$

and $\lim_{n\to\infty} b_n = 0 \Rightarrow \sum (-1)^{n+1} b_n$ converges by the Alternating series test.

11. For what values of p is the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ convergent?

If $p > 0$, $\frac{1}{(n+1)^p} \leq \frac{1}{n^p}$ and $\lim_{n\to\infty} \frac{1}{n^p} = 0$

$\Rightarrow$ converges by the alternating series test.

If $p \leq 0$, $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ Does not exist

$\Rightarrow$ the series diverges.

$\Rightarrow$ a_n converges $\Leftrightarrow$ $p > 0$

12. Determine whether the series is absolutely convergent or conditionally convergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$

(a) $b_n = \frac{1}{\sqrt{n}} > 0$ for $n \geq 1$ b_n decreasing for $n \geq 1$, $\lim_{n \to \infty} b_n = 0$

$\Rightarrow \sum \frac{(-1)^{n-1}}{\sqrt{n}}$ Converges by alternating series test.

But $\sum \frac{1}{\sqrt{n}}$ diverges because p-series

$\Rightarrow \sum \frac{(-1)^{n-1}}{\sqrt{n}}$ conditionaly convergent.

(b) $0 < \left| \frac{\sin n}{2^n} \right| < \frac{1}{2^n}$; $n \geq 1$ & $\sum \frac{1}{2^n}$ conv. (geometric series)

$\Rightarrow \sum \left| \frac{\sin n}{2^n} \right|$ conv. by comparison test

$\Rightarrow \sum \frac{\sin n}{2^n}$ conv. absolutely

13. Use the ratio test to determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$

(a) $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n}\right| = \lim \left|\frac{1}{5} \cdot \frac{n+1}{n}\right|$

$\displaystyle = \frac{1}{5} \Rightarrow$ Abs. Conv.

(b) $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{\cos\left(\frac{(n+1)\pi}{3}\right)}{(n+1)!} \cdot \frac{n!}{\cos(n\pi/3)}\right|$

$\displaystyle = \lim \left|\frac{\cos(n+1)\pi/3}{(n+1)\cos n\pi/3}\right| = \lim \frac{c}{n+1} = 0 < 1$

$\Rightarrow$ abs. conv.

14. For which of the following series is the ratio test inconclusive (that is, it fails to give a definite answer)?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\qquad \displaystyle\lim \frac{n^3}{(n+1)^3} = 1 \Rightarrow$ inconclusive

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ $\qquad \displaystyle\lim \frac{n+1}{2^n} \cdot \frac{2^{n+1}}{n} = \lim \frac{n+1}{2n} = \frac{1}{2} \Rightarrow$ conclusive (conv.)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$ $\qquad \displaystyle\lim \frac{(-3)^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-3)^{n-1}} = \lim 3\sqrt{\frac{n}{n+1}} = 3 \Rightarrow$ conclusive (div.)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$ $\qquad \displaystyle\lim \frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}} = 1 \Rightarrow$ inconclusive

For all the questions above, compute $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$