1. Find the equation of the tangent line to the curve at the given point.

   (a) $y = 4x - 3x^2$ at $(2, -4)$

   (b) $y = x^3 - 3x + 1$ at $(2, 3)$

   (c) $y = \sqrt{x}$ at $(1, 1)$

   (d) $y = \dfrac{2x+1}{x+2}$ at $(1, 1)$

Things to know: 1. Equation of a line: $y - y_0 = m(x - x_0)$

2. Slope of the tangent line = derivative of f at x_0

$m = f'(x_0)$

b) $y = x^3 - 3x + 1$ at $(2, 3)$

$y' = 3x^2 - 3$

$m = f'(x_0)$

$f'(2) = 3(2)^2 - 3 = 12 - 3 = \boxed{9}$

$y - 3 = 9(x - 2)$

$y - 3 = 9x - 18 \Rightarrow y = 9x - 15$

d) $y = \dfrac{2x+1}{x+2}$ at $(1, 1)$

Quotient Rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

$y' = \dfrac{2(x+2) - (2x+1)}{(x+2)^2}$

$f'(1) = \dfrac{2(1+2) - (2+1)}{(1+2)^2}$

$= \dfrac{2 \times 3 - 3}{3^2} = \dfrac{3}{9} = \dfrac{1}{3}$

$y - 1 = \dfrac{1}{3}(x - 1)$

$y = \dfrac{1}{3}x + \dfrac{2}{3}$

a) $y = 4x - 8x^2$ at $(2, -4)$

$y' = 4 - 6x$ $\quad f'(2) = 4 - 12 = -8$

$\Rightarrow y + 4 = -8(x - 2)$

$\Rightarrow y = -8x + 12$

c) $y = \sqrt{x}$ at $(1, 1)$

$y' = \dfrac{1}{2\sqrt{x}}$ $\quad f'(1) = \dfrac{1}{2\sqrt{1}} = \dfrac{1}{2}$

$\Rightarrow y - 1 = \dfrac{1}{2}(x - 1)$

$\Rightarrow y = \frac{1}{2}x + \frac{1}{2}$

2. Find the derivative of the function using the definition of derivative.

(a) $f(x) = 3x - 8$

→ (b) $g(t) = \dfrac{1}{\sqrt{t}}$

Things to Know: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

b) $g(t) = \dfrac{1}{\sqrt{t}}$ $\qquad g'(t)$

$$g'(t) = \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h}$$

$(a-b)(a+b) = a^2 - b^2$

$$= \lim_{h\to 0} \frac{\frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t}\,\sqrt{t+h}} \times \frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(\sqrt{t})^2 - (\sqrt{t+h})^2}{\sqrt{t}\,\sqrt{t+h}\,(\sqrt{t} + \sqrt{t+h})}}{h} = \lim_{h\to 0} \frac{t - (t+h)}{h\sqrt{t}\,\sqrt{t+h}\,(\sqrt{t} + \sqrt{t+h})}$$

$$= \lim_{h\to 0} \frac{\cancel{t} - \cancel{t} - h}{h\sqrt{t}\,\sqrt{t+h}\,(\sqrt{t} + \sqrt{t+h})} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}\sqrt{t}\,\sqrt{t+h}\,(\sqrt{t} + \sqrt{t+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{t}\,\sqrt{t+h}\,(\sqrt{t} + \sqrt{t+h})} = \frac{-1}{\sqrt{t}\,\sqrt{t}\,(\sqrt{t} + \sqrt{t})} = \frac{-1}{2t\sqrt{t}}$$

$$g(t) = \frac{1}{\sqrt{t}} = \frac{1}{t^{1/2}} = t^{-1/2}$$

$$g'(t) = -\tfrac{1}{2}\, t^{-1/2 - 1} = -\frac{1}{2}\, t^{-3/2}$$

a) $f(x) = 3x - 8$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h) - 8 - (3x-8)}{h}$$

$$= \lim_{h \to 0} \frac{3x + 3h - 8 - 3x + 8}{h}$$

$$= \lim_{h \to 0} \frac{3h}{h} = 3$$

3. Differentiate the following functions using the differentiation rules.

→ (a) $f(x) = e^x - x^5$

(b) $f(x) = \dfrac{x^2 + x - 2}{x^3 + 6}$

(c) $f(x) = \left(x^3 + 1\right) e^x$

(d) $f(x) = \left(5x^6 + 2x^3\right)^4$

→ (e) $f(x) = (3x-1)^4(2x+1)^{-3}$

Things to Know:

1. $(x^n)' = nx^{n-1}$
2. $(e^x)' = e^x$
3. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'g - fg'}{g^2}$
4. $(f(x) \cdot g(x))' = f'g + fg'$
5. $(f(g(x)))' = f'(g) \cdot g'$

a) $f(x) = e^x - x^5$

$f'(x) = e^x - 5x^4$

e^{2x} $\quad u = 2x \to u' = 2$

$f = e^u \quad f' = e^u \quad 2e^{2x}$

e) $f(x) = \underbrace{(3x-1)^4}_{g(x)}\underbrace{(2x+1)^{-3}}_{h(x)}$

$i(x) = 2x+1$
$j(x) = x^{-3}$ $\Big\}\ h(x) = j(i(x))$

$g(x) = (3x-1)^4$

$k(x) = 3x-1$
$l(x) = x^4$

$g(x) = l(k(x))$

$g'(x) = 4(3x-1)^3 \cdot 3 = 12(3x-1)^3$

$h'(x) = -3(2x+1)^{-4} \cdot 2 = -6(2x+1)^{-4}$

$f'(x) = g'h + gh'$

$= 12(3x-1)^3(2x+1)^{-3} + (3x-1)^4 \cdot (-6(2x+1)^{-4})$

b) $f(x) = \dfrac{x^2+x-2}{x^3+6}$

$$f'(x) = \frac{(2x+1)(x^3+6) - 3x^2(x^2+x-2)}{(x^3+6)^2}$$

c) $f(x) = (x^3+1)e^x$

$$f'(x) = 3x^2e^x + (x^3+1)e^x$$

d) $f(x) = (5x^6+2x^3)^4$

$$f'(x) = 4(5x^6+2x^3)^3(30x^5+6x^2)$$

4. Differentiate the following functions.

(a) $f(x) = x^2 \sin(7x)$

(b) $f(\theta) = \sec(\theta)\tan(4\theta)$

(c) $f(t) = \dfrac{t\sin(t^2+2t)}{1+t}$

(d) $f(x) = \dfrac{\sin(4x^3)}{1+\tan(x^2)}$

Things to Know :

1. $(\sin x)' = \cos x$
2. $(\cos x)' = -\sin x$
3. $(\tan x)' = \sec^2 x$
4. $(\csc x)' = -\csc x \cot x$
5. $(\sec x)' = \sec x \tan x$
6. $(\cot x)' = -\csc^2 x$

c) $f(t) = \dfrac{t\sin(t^2+2t)}{1+t}$

$$f'(t) = \frac{\left(\sin(t^2+2t) + t\cos(t^2+2t)(2t+2)\right)\cdot(1+t) - t\sin(t^2+2t)}{(1+t)^2}$$

d) $f(x) = \dfrac{\sin(4x^3)}{1+\tan(x^2)}$

$$f'(x) = \frac{(\cos(4x^3)12x^2)(1+\tan(x^2)) - \sec^2(x^2)\cdot 2x\sin(4x^3)}{(1+\tan(x^2))^2}$$

a) $f'(x) = 2x\sin(7x) + x^2\cos(7x)\cdot 7$

b) $f'(\theta) = \sec(\theta)\tan(\theta)\tan(4\theta) + \sec(\theta)\sec^2(4\theta)\cdot 4$

5. Find the limit.

(a) $\lim\limits_{x\to 0} \dfrac{\sin(5x)}{3x}$

(b) $\lim\limits_{x\to 0} \dfrac{\sin(x)}{\sin(\pi x)}$

(c) $\lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x}$

$$* \quad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

a) $$\lim_{x\to 0} \frac{\sin(5x)}{3x} \cdot \frac{5x}{5x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \cdot \frac{5x}{3x}$$

$$= \underbrace{\lim_{x\to 0} \frac{\sin 5x}{5x}} \cdot \underbrace{\lim_{x\to 0} \frac{5\cancel{x}}{3\cancel{x}}}$$

$$= 1 \cdot \frac{5}{3} = \frac{5}{3}$$

c) $$\lim_{x\to 0} \frac{\sin(x^2)}{x} = \lim_{x\to 0} \frac{\sin(x^2)}{x^2} \cdot \frac{x^2}{x}$$

$$= \underbrace{\lim_{x\to 0} \frac{\sin(x^2)}{x^2}} \cdot \lim_{x\to 0} \frac{x^2}{x}$$

$$= 1 \cdot \lim_{x\to 0} x = 1 \times 0 = 0$$

b) $$\lim_{x\to 0} \frac{\sin x}{\sin(\pi x)} = \lim \frac{\sin x}{x} \cdot \frac{x}{\pi x} \cdot \frac{1}{\frac{\sin \pi x}{\pi x}}$$

$$= \frac{1}{\pi}$$

6. If $F(x) = f(3x)$, where $f'(0) = 2$, find $F'(0)$.

$F(x) = f(3x)$

$= f(g(x))$

$g(x) = 3x \rightarrow g'(x) = 3$

$g(0) = 0$

$F'(x) = f'(g(x)) \cdot g'(x)$

$F'(0) = f'(0) \cdot g'(0) = 2 \times 3 = 6$

7. Find the 50th derivative of $y = \cos(x)$.

0 $y = \cos(x)$

1 $y' = -\sin(x)$

2 $y'' = -\cos(x)$

3 $y''' = \sin(x)$

4 $y^{(5)} = \cos(x)$

$$\begin{array}{r|l} 50 & 4 \\ \hline 4 & 12 \\ \hline 10 & \\ 8 & \\ \hline 2 & \end{array}$$

$50 = 12 \times 4 + 2$

$y^{(50)}(x) = y'' = -\cos x$

8. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

(a) $y\sin(2x) = x\cos(2y)$ at $(\frac{\pi}{2}, \frac{\pi}{4})$

(b) $x^2 + 2xy + 4y^2 = 12$ at $(2, 1)$

a) $\frac{d}{dx}\left(y \sin(2x) = x \cos(2y)\right)$ $\qquad y' = \frac{d}{dx} y$

$$\frac{d}{dx}(y)\sin(2x) + y\frac{d}{dx}(\sin(2x)) = \frac{d}{dx}(x)\cos(2y) + x\frac{d}{dx}\cos(2y)$$

$$\rightarrow y'\sin(2x) + y\cos(2x)\cdot 2 = \cos(2y) + x(-\sin(2y))\cdot 2y'$$

$$y'\sin(2x) + 2x\sin(2y)\,y' = -2y\cos(2x) + \cos(2y)$$

$$y'(\sin(2x) + 2x\sin(2y)) = -2y\cos(2x) + \cos(2y)$$

$$y' = \frac{-2y\cos(2x) + \cos(2y)}{\sin(2x) + 2x\sin(2y)} \qquad \text{at } \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$$

$$y' = \frac{-\frac{\pi}{2}\cos(\pi) + \cos(\frac{\pi}{2})}{\sin(\pi) + \pi\sin(\frac{\pi}{2})} = \frac{-\frac{\pi}{2}(-1) + 0}{0 + \pi} = \frac{\pi/2}{\pi} = \boxed{1/2}$$

$$y - y_0 = m(x - x_0)$$

$$y - \frac{\pi}{4} = \frac{1}{2}\left(x - \frac{\pi}{2}\right)$$

$$y = \frac{1}{2}x$$

b) $\frac{d}{dx}\left(x^2 + 2xy + 4y^2 = 12\right)$ at $(2,1)$ $y' = \frac{dy}{dx}$

$y = f(x)$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) + \frac{d}{dx}(4y^2) = \frac{d}{dx}(12)$ $y^2 = (f(x))^2$

$2x + \frac{d}{dx}(2x)y + 2x\frac{d}{dx}y + 4 \times 2y \cdot \frac{d}{dx}y = 0$ $2f(x) \cdot f'(x)$

$2x + 2y + 2xy' + 8yy' = 0$ $(2,1)$

$4 + 2 + 4y' + 8y' = 0$

$6 + 12y' = 0 \Rightarrow y' = \frac{-1}{2}$

$y - 1 = \frac{-1}{2}(x - 2)$

9. Find y'.

(a) $y = \sqrt{x} \ln x$

(b) $y = \ln(1 + \ln x)$

$$(\ln(x))' = \frac{1}{x}$$

a) $y = \sqrt{x} \ln(x)$ $\sqrt{x} = x^{1/2} \rightarrow \frac{1}{2} x^{1/2 - 1}$

$$y' = \frac{1}{2\sqrt{x}} \cdot \ln(x) + \sqrt{x} \cdot \frac{1}{x}$$

b) $y = \ln(1 + \ln x)$

$$y' = \frac{1}{1 + \ln x} \cdot \frac{1}{x} = \frac{1}{x(1 + \ln x)} = \frac{1/x}{1 + \ln(x)}$$

$P(0)$ $t=1$

10. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420. $\rightarrow P(1)$

    (a) Find an expression for the number of bacteria after t hours.

    (b) Find the rate of growth after 3 hours.

$P(t) = P(0)\, e^{kt}$

$P(1) = 420$

$P(0) = 100$

$420 = 100 \cdot e^{K} \Rightarrow e^{K} = 4.2$

$\Rightarrow \ln(e^{K}) = \ln(4.2)$

$\Rightarrow K = \ln(4.2)$

a) $P(t) = 100\, e^{\ln(4.2)\, t}$

b) $P'(t) = 100 \cdot \ln(4.2)\, e^{\ln(4.2)\, t}$

$P'(3) = 100 \ln(4.2)\, e^{3\ln(4.2)}$