Sequences and Series
Quantitative Learning Center at the University of Connecticut

Sequences
A sequence is a list of numbers \(\{a_1, a_2, a_3, \ldots\}\). The sequence converges if \(\lim_{n \to \infty} a_n\) exists. If the limit does not exist, then the sequence diverges.

Series
A series \(\sum_{n=1}^{\infty} a_n\) is the limit of the partial sums \(s_n = \sum_{i=1}^{n} a_i\).

1. If \(\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{n-1} = \sum_{n=1}^{\infty} a_{n+1}\), Write a few terms to see why.

2. The harmonic series is the sum of 1/n.

3. The geometric series is \(\sum_{n=1}^{\infty} r^n\).

4. The series converges if \(|r| < 1\) and diverges if \(|r| \geq 1\).

5. The alternating series test.

6. The alternating harmonic series is \(\sum_{n=1}^{\infty} (-1)^{n-1} a_n\).

7. The integral test.

8. The root test.

Comparison Test
Suppose all \(a_n\) are positive and all \(b_n\) in another sequence are also positive.

1. If \(\sum_{n=1}^{\infty} a_n\) converges, then \(\sum_{n=1}^{\infty} b_n\) converges.

2. If \(\sum_{n=1}^{\infty} b_n\) diverges, then \(\sum_{n=1}^{\infty} a_n\) diverges.

Direct Comparison
If \(0 < a_n \leq b_n\), then \(\sum_{n=1}^{\infty} a_n\) converges and \(\sum_{n=1}^{\infty} b_n\) converges.

Limit Comparison Test
If \(\lim_{n \to \infty} \frac{a_n}{b_n} = L\), then \(\sum_{n=1}^{\infty} a_n\) converges if \(\sum_{n=1}^{\infty} b_n\) converges.

Absolute Convergence, Root, and Ratio Tests

Absolute Convergence
A series \(\sum_{n=1}^{\infty} a_n\) converges absolutely if \(\sum_{n=1}^{\infty} |a_n|\) converges.

Root Test
If \(\lim_{n \to \infty} \sqrt[n]{|a_n|} = L \leq 1\), then \(\sum_{n=1}^{\infty} a_n\) converges absolutely.

Ratio Test
If \(\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L < 1\), then \(\sum_{n=1}^{\infty} a_n\) converges absolutely.

Convergence Test Strategies
1. Classify the form of the series.
2. Compare it to a known series.
3. Use absolute convergence.
4. Use the root test if possible.
5. Use the ratio test if possible.
6. Use the alternating series test if possible.
7. Use the direct comparison test if possible.
8. Use the limit comparison test if possible.

Note:
1. If \(a_n = 0 \rightarrow 0\), then \(\sum_{n=1}^{\infty} a_n\) diverges.
2. If the series is of the form \(\sum_{n=1}^{\infty} a_n r^{n-1}\) or \(\sum_{n=1}^{\infty} a_n r^n\) then use the geometric series.
3. If the series is \(\sum_{n=1}^{\infty} \frac{1}{n}\), then use the harmonic series.
4. If \(a_n = f(n)\) and \(f(x)\) is easy to integrate, try integral test.
5. If the series has either form \(\sum_{n=1}^{\infty} (-1)^n a_n\) or \(\sum_{n=1}^{\infty} (-1)^n b_n\) with \(b_n > 0\), then try the alternating series test.
6. If \(a_n\) diverges like \(b_n\) and \(\sum_{n=1}^{\infty} b_n\) is known, use limit comparison test (positive terms).
7. If \(a_n = 0\) and \(\sum_{n=1}^{\infty} b_n\) converges, use comparison and absolute convergence.
8. If the series has \(1/n^r\) power, try the ratio or root tests.